Even-Odd Set Partitions, Saddle-Point Method and Wyman Admissibility

Bruno Salvy

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Even-Odd Set Partitions, Saddle-Point Method and Wyman Admissibility

Bruno SALVY

N ° 4201
Juin 2001

THÈME 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4201--FR+ENG

Even-Odd Set Partitions,
Saddle-Point Method
and Wyman Admissibility

Bruno SALVY

Thème 2 — Génie logiciel
et calcul symbolique
Projet Algo

Rapport de recherche n ° 4201 — Juin 2001 — 14 pages

Abstract: The reciprocal of the generating function of the Bell numbers enumerates the difference between numbers of set partitions with even and odd number of blocks. The asymptotic behaviour of this oscillatory sequence is obtained by a saddle-point analysis involving two conjugate saddle points. Technical details of the analysis are dealt with by appealing to Wyman's class of admissible functions.

Key-words: Bell numbers, asymptotics, Wyman admissibility

(Résumé : tsvp)

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33) 01 39 63 55 11 – Télécopie : (33) 01 39 63 53 30

Partitions d'ensembles paires-impaires, méthode du col et admissibilité au sens de Wyman

Résumé : L'inverse de la série génératrice des nombres de Bell compte la différence entre le nombre de partitions d'ensembles à nombre de blocs impair et le nombre de celles dont le nombre de blocs pair. Le comportement asymptotique de cette suite oscillante est obtenu par une analyse de col faisant intervenir deux points cols conjugués. Les détails techniques de l'analyse sont réglés en invoquant la classe des fonctions admissibles de Wyman.

Mots-clé : Nombres de Bell, asymptotique, admissibilité de Wyman

EVEN-ODD SET PARTITIONS, SADDLE-POINT METHOD AND WYMAN ADMISSIBILITY

BRUNO SALVY

ABSTRACT. The reciprocal of the generating function of the Bell numbers enumerates the difference between numbers of set partitions with even and odd number of blocks. The asymptotic behaviour of this oscillatory sequence is obtained by a saddle-point analysis involving two conjugate saddle points. Technical details of the analysis are dealt with by appealing to Wyman's class of admissible functions.

Wyman introduced in [19] a class of functions for which the saddle-point method yields an asymptotic expansion of the Laurent coefficients when the index tends to infinity. While less well-known to combinatorialists than Hayman's or Harris & Schoenfeld's classes, Wyman admissibility applies to a larger class of functions. The drawback is that its application is less straightforward. In this article, we give a detailed application of Wyman's method on a combinatorial problem posed to us by N. Calkin who got it from H. Wilf.

The Bell number B_n counts the number of partitions of a set with n distinct elements into nonempty subsets. The first ten elements of this sequence are

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975.$$

This classical sequence appears as A000110 in the EIS [17], where an extensive bibliography can be found. From classical combinatorics (see for instance [2, 4]), the exponential generating function of this sequence is

$$B(z) = \sum_{n\geq 0} B_n \frac{z^n}{n!} = \exp(e^z - 1). \tag{1}$$

The asymptotic expansion of the Bell numbers is a classical example for the *saddle-point method* in combinatorics, treated in great detail by N. G. de Bruijn in [6]. The main steps of the method are recalled in Section 1 below.

The exponential in (1) stems from the classical translation of combinatorial sets. If instead one is interested in numbers of partitions with even (resp. odd) numbers of parts, the generating function becomes $\cosh(e^z - 1)$ (resp. $\sinh(e^z - 1)$). The extent to which these numbers differ is embodied by the generating function

$$S(z) = \sum_{n\geq 0} S_n \frac{z^n}{n!} := \cosh(e^z - 1) - \sinh(e^z - 1) = \exp(1 - e^z).$$

The coefficient S_n in this series is the number of partitions of a set of size n in an even number of parts minus the number of partitions with an odd number of parts. The first few numbers are

$$1, -1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180.$$

The question of the asymptotic behaviour of this sequence (and thus in particular an explanation of its sign pattern) was raised by H. Wilf. We give an answer in Theorem 1 below.

This sequence, with signs removed, appears as A000587 in the EIS, together with a few references. In particular, Uppuluri and Carpenter derive in [18] the analogue of Dobinsky's formula and various other relations by either paralleling the classical derivations for the Bell numbers or exploiting the fact that both series are reciprocal of each other. Some of these results had been derived previously by Beard [1], in the *Journal of the Institute of Actuaries*, where the asymptotic expansion of S_n has also been considered, "although of limited actuarial application". His derivation is very formal and he admits that his article "has been largely relieved of various analytical considerations which arise at a number of points". Our aim is to show that a rigourous analytic derivation is now possible without too much work, thanks to Wyman's general results in [19] which were not available to Beard.

Our main result is the following.

Theorem 1. *Let $S_n/n!$ be the coefficient of z^n in the Taylor series of $\exp(1-e^z)$ at the origin. Then there exist sequences A_n, ϕ_n and u_n such that, as $n\to\infty$,*

$$S_n = A_n\cos(\phi_n) + O(u_n),$$

where $\log(u_n/A_n) = o(n/\log^k n)$, for all $k>0$,

$$\log\frac{A_n}{n!} = n\left(-\log\log n + \frac{\log\log n + 1}{\log n} + \frac{(\log\log n)^2-\pi^2}{2\log^2 n}\right.$$
$$\left. + \frac{2(\log\log n)^3 - 3(\log\log n)^2 - 6\pi^2\log\log n + 3\pi^2}{6\log^3 n} + O\left(\frac{(\log\log n)^4}{\log^4 n}\right)\right),$$

$$\phi_n = \pi n\left(\frac{1}{\log n} + \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n - \pi^3/3}{\log^3 n} + O\left(\frac{(\log\log n)^3}{\log^4 n}\right)\right).$$

In particular, $\log B_n/\log A_n \to 1$ and $B_n/A_n\to\infty$.

Figure 1 shows the values of $\log(B_n/n!)$ (all values are negative since $B_n \le n!$), of $\log|S_n/n!|\operatorname{sign}(S_n)$ (of larger absolute value than the previous one) and $|T_n|\operatorname{sign}(T_n)$ where T_n is obtained by multiplying the first four terms of the asymptotic expansion of $\log(A_n/n!)$ above by the cosine of the first 5 terms of the expansion of ϕ_n. This latter sequence appears to have a slightly larger absolute value than the previous one and a very similar sign pattern.

1. Saddle-Point Method

We first recall the main steps of the calculations performed during the saddle-point method. The starting point is Cauchy's integral formula giving the nth Taylor coefficient f_n of an analytic function $f(z)$ at the origin as:

$$f_n = \frac{1}{2i\pi}\oint f(z)\frac{dz}{z^{n+1}}, \tag{2}$$

where the contour is for instance a circle enclosing the origin and no other singularity of f. A nice heuristic explanation of the saddle-point method for the asymptotics of integrals can be found in [16]. On Figure 2, the function $|B(z)/z^{n+1}|$ is displayed for $n=10$. (Colours indicate the argument and values higher than 1 have been removed to keep the picture finite.) The idea of the method is to choose a contour

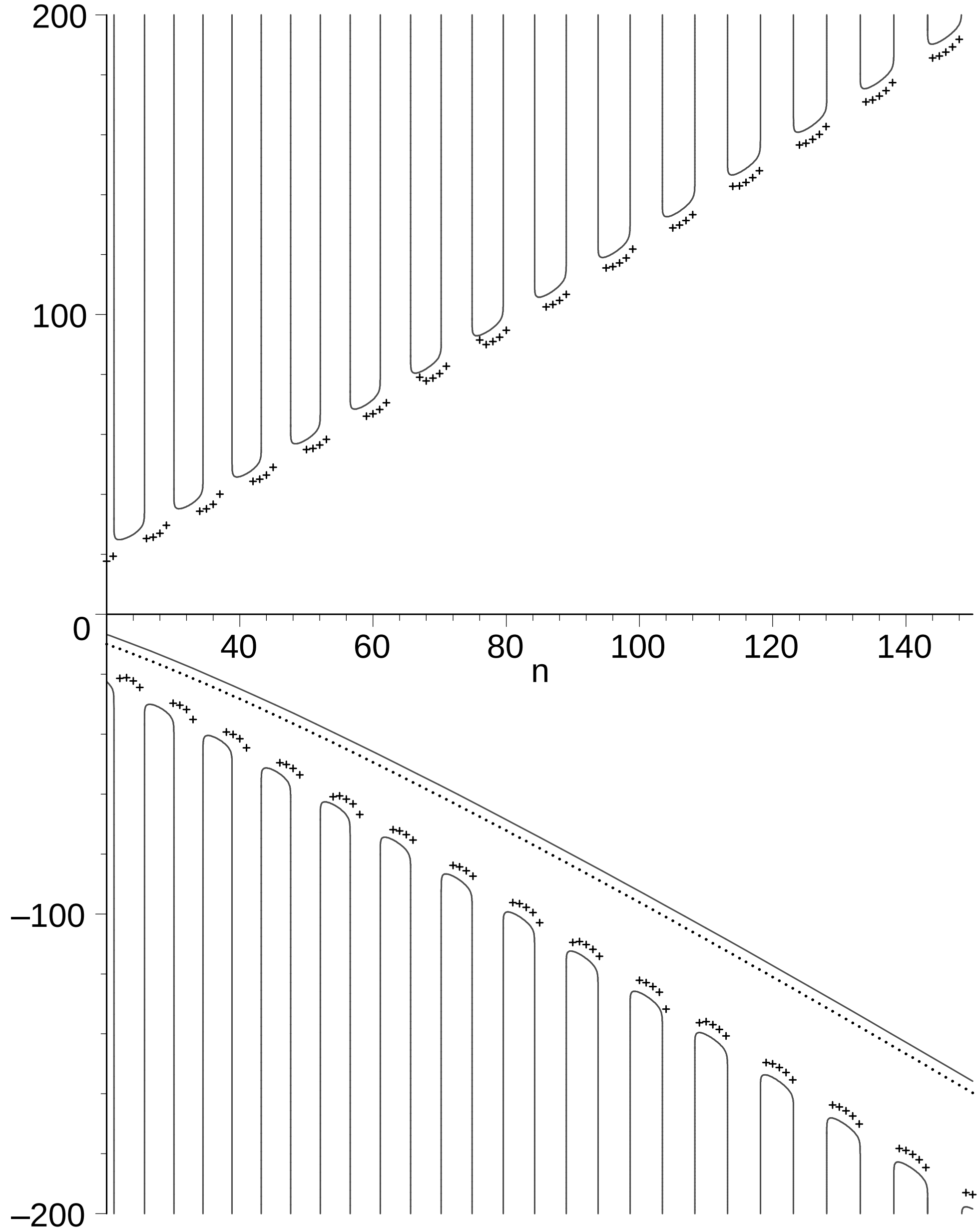

FIGURE 1. Actual values vs. asymptotic estimates: $\log(B_n/n!)$ (dots), its asymptotic estimate (continuous curve), $\log|S_n/n!|\,\mathrm{sign}(S_n)$ (crosses), its asymptotic estimate (discontinuous curve)

passing through the saddle point. As n becomes large, a small portion of the circle on both sides of the saddle point contributes most of the integral. Locally, the integrand behaves as a Gaussian function and the error made in approximating it by an actual Gaussian function becomes exponentially small as n increases. This leads to a three-step formal process:

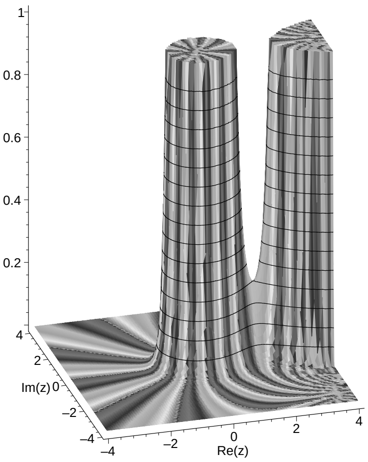

FIGURE 2. $|B(z)/z^{11}|$

1. Locate the saddle point. This point is given as a zero of the derivative (or more conveniently the logarithmic derivative) of the integrand. The equation to be solved has the form

$$r\frac{f'(r)}{f(r)} = n+1. \tag{3}$$

In all cases the left-hand side does not depend on n. In general, this equation does not have a closed-form solution. This introduces technicalities that involve computing an asymptotic expansion of r as n tends to infinity;

2. Change the variables locally to make the function Gaussian. Introducing a new variable u, this is achieved by the change of variables

$$\frac{f(z)}{z^{n+1}} = \frac{f(r)}{r^{n+1}}\exp(-u^2/2). \tag{4}$$

The integral (2) then becomes

$$f_n = \frac{1}{2i\pi}\frac{f(r)}{r^{n+1}}\oint \exp(-u^2)\frac{dz}{du}\,du. \tag{5}$$

The expansion of the derivative dz/du is obtained from (4). Letting $h(z)$ denote $\log(f(z)/z^{n+1})$, this equation rewrites

$$h(z) = h(r) - \frac{u^2}{2}. \tag{6}$$

In view of the definition of the saddle point by $h'(r)=0$, this leads to

$$iu = \sqrt{h''(r)}(z-r) + \frac{h^{(3)}(r)}{\sqrt{h''(r)}}\frac{(z-r)^2}{6} + \cdots.$$

Inverting this expansion and differentiating with respect to u leads to

$$\frac{dz}{du} = \frac{i}{\sqrt{h''(r)}} + \frac{h^{(3)}(r)}{h''(r)^2}\frac{u}{3} + \frac{6h^{(4)}(r)h''(r) - 10h^{(3)}(r)^2}{h''(r)^{7/2}}\frac{iu^2}{48} + \cdots. \tag{7}$$

3. Approximate the integral locally by an integral from $-\infty$ to ∞. Injecting the expansion above in (5) and using the classical value of

$$\int_{-\infty}^{\infty} u^{2k} e^{-u^2/2}\, du = \sqrt{2\pi} \cdot 1 \cdot 3 \cdots (2k-1), \qquad k \in \mathbb{N}$$

thus leads to the approximation

$$f_n = \frac{f(r)}{r^{n+1}\sqrt{2\pi h''(r)}} \left(1 + \frac{6h^{(4)}(r)h''(r) - 10h^{(3)}(r)^2}{48h''(r)^3} + \cdots\right). \tag{8}$$

The rest of this section is devoted to working out these formal manipulations for the cases of the numbers B_n and S_n from the introduction. These computations motivate the result of Theorem 1, which is proved by the study and validation of this formal process from an analytic point of view in Section 2. Of course, the results we obtain for B_n are not new, but the sequence S_n is so intimately related to B_n that the derivations cannot be separated.

1.1. **Location of the Saddle Point.** In the case of the Bell numbers, using the saddle-point equation (3), the saddle-point ρ satisfies

$$\rho \exp(\rho) = n + 1. \tag{9}$$

This equation can be solved explicitly in terms of the Lambert W function [5]. It has one root of smallest modulus, which is real positive and whose asymptotic behaviour can be found by rewriting the equation as

$$\rho = \log(n+1) - \log(\rho)$$

and iterating [6]. This leads to

$$\rho = \log n - \log\log n + \frac{\log\log n}{\log n} + \frac{\log\log n(\log\log n - 2)}{2\log^2 n} + \cdots. \tag{10}$$

A nice observation of Comtet's [3] is that the coefficients in the polynomials in $\log\log n$ appearing in the numerators of this expansion are Stirling numbers of the first kind. This observation can be generalized and has consequences on the efficient computation of a family of asymptotic expansions, see [14].

For the sequence S_n, the corresponding saddle-point equation reads

$$-r\exp(r) = n + 1, \tag{11}$$

so that there are two solutions of smallest modulus R_ϵ (with $\epsilon = \pm 1$), close to $\rho + \epsilon i\pi$ (see Figure 3). Again, these solutions can be expressed using different branches of the Lambert W function. The asymptotic expansions of these solutions are obtained by setting $R_\epsilon = \rho + \epsilon i\pi + u$. Injecting this into the saddle-point equation and rewriting in view of (9) leads to

$$\rho^{-1} = \frac{e^{-u} - 1}{u + \epsilon i\pi}.$$

Power series reversion then yields the asymptotic expansion of the saddle points R_ϵ as $n \to \infty$:

$$R_\epsilon = \rho + \epsilon i\pi - \epsilon i\pi\rho^{-1} + (\epsilon i\pi - \pi^2/2)\rho^{-2} + \cdots. \tag{12}$$

1.2. **Local Expansion and Termwise Integration.** In the case of the Bell numbers the function $h(z)$ from Equation (6) satisfies

$$h^{(k)}(z) = e^z + (-1)^k (n+1)\frac{(k-1)!}{z^{k+1}}, \qquad k = 1, 2, \dots .$$

Further simplification occurs when $z = \rho$ in view of (9) and thus the final resulting approximation (8) becomes

$$\frac{B_n}{n!} = \frac{\exp(e^\rho - \frac{\rho}{2} - 1)}{\rho^{n+1}\sqrt{2\pi(1+\rho^{-1})}}\left(1 + e^{-\rho}\frac{2\rho^4 - 3\rho^3 - 20\rho^2 - 18\rho + 2}{24\rho(\rho+1)^3} + \cdots\right). \tag{13}$$

That this approximation provides an asymptotic expansion of the Bell numbers is proved in Section 2.

We also note that this expansion in terms of ρ cannot be easily converted into an expansion in terms of n. Indeed, from the saddle-point equation (9) we get that $\exp(\rho) = (n+1)/\rho$ so that all the terms in the asymptotic expansion of $\exp(\rho)$ tend to infinity. It is therefore impossible to compute another level of exponential from this result. However, an expansion for $\log(B_n/n!)$ is available from (13). In terms of ρ first, this is

$$\log\frac{B_n}{n!} = e^\rho(-\rho\log\rho + 1) - \frac{\rho}{2} - 1 - \frac{1}{2}\log 2\pi + O(\rho^{-1}). \tag{14}$$

When replacing ρ by its expansion, only the first term remains and we get

$$\log\frac{B_n}{n!} = n\left(-\log\log n + \frac{\log\log n + 1}{\log n} + \frac{(\log\log n)^2}{2\log^2 n} + \frac{(\log\log n)^2(2\log\log n - 3)}{6\log^3 n} + O\left(\frac{(\log\log n)^4}{\log^4 n}\right)\right), \tag{15}$$

where we have used the fact that differences between n and $n+1$ are absorbed by the $O()$ term. This is exactly the classical result as given for instance in [6, Eq. (6.2.7) p. 108].

The computation for S_n is very similar to the previous one: the corresponding function h now satisfies

$$h^{(k)}(z) = -e^z + (-1)^k (n+1)\frac{(k-1)!}{z^{k+1}}, \qquad k = 1, 2, \dots .$$

When $z = R_\epsilon$, simplifications are induced by (11) and thus application to each saddle point of the formal process described above yields

$$C_\epsilon := \frac{\exp(-e^{R_\epsilon} - \frac{R_\epsilon}{2} + 1 + \frac{\epsilon i\pi}{2})}{R_\epsilon^{n+1}\sqrt{2\pi(1+R_\epsilon^{-1})}}\left(1 + e^{-R_\epsilon}\frac{2R_\epsilon^4 - 3R_\epsilon^3 - 20R_\epsilon^2 - 18R_\epsilon + 2}{24R_\epsilon(R_\epsilon+1)^3} + \cdots\right). \tag{16}$$

The final result is simply $C_1 + C_{-1}$. Modulus and phase are calculated by considering the real and imaginary parts of $\log C_\epsilon$. An asymptotic expansion of these is obtained using the expansion (12) of R_ϵ in terms of ρ. The computation requires only the first part of (16), namely

$$\log C_\epsilon = -e^{R_\epsilon}(-R_\epsilon\log R_\epsilon + 1) + O(R_\epsilon),$$

where again we have used the saddle-point equation (11) to rewrite $n+1$. Expressing the right-hand side in terms of ρ, we finally get the following expansions which should be compared to that of $\log B_n/n!$ in (14)

$$\Re \log C_\epsilon = e^\rho(-\rho\log\rho + 1 - \frac{\pi^2}{2\rho} + \frac{\pi^2}{2\rho^2} + \frac{\pi^4 - 2\pi^2}{4\rho^3} + O(\rho^{-4})),$$
$$\Im \log C_\epsilon = \epsilon e^\rho(-\pi + \frac{\pi^3}{3\rho^2} - \frac{5\pi^3}{6\rho^3} + \frac{15\pi^3 - 2\pi^5}{10\rho^4} + O(\rho^{-5})).$$

We finally get the approximation

$$\frac{S_n}{n!} \approx \exp(\Re \log C_\epsilon)\cos(\Im \log C_\epsilon).$$

The "asymptotic amplitude" of the numbers S_n is $A_n = n!\exp(\Re\log C_\epsilon)$. We get the asymptotic behaviour in Theorem 1 by injecting the expansion of ρ from (10). The same computations for the "asymptotic phase" $\phi_n := \log C_1$ conclude the computational part of the proof of Theorem 1.

As one can see in this section, computations in this area are usually tedious and better done using a computer algebra system. However, this requires a system that can handle complicated asymptotic scales as well as asymptotic functional inversion. Computer algebra algorithms for this purpose are described in [15], together with a combinatorial example.

2. Wyman admissibility

The computations performed in the saddle-point method require analytic justification. It is well known that an asymptotic expansion obtained formally can be badly wrong, and Olver gives a striking example in [13, p. 76–79]. In particular, the saddle-point method requires the function to have exponential increase in the neighbourhood of its smallest singularity (or at infinity if it is entire). In other cases, one should use Darboux's method or singularity analysis [7]; see [11] for a survey of available methods.

Three approximations are performed that may introduce error terms of an order larger than the terms of the final expansion (8): an expansion (7) which is valid only locally; approximation of the integral by a full Gaussian integral; termwise integration.

The first two approximations are usually legitimated by cutting the contour in several pieces: it is necessary to determine a neighbourhood of the saddle point(s) sufficiently large for the rest of the integral to be negligible compared to the terms in (8) and sufficiently small for the local expansion of dz/du to be convergent. Then, in the integrals on neighbourhoods of the saddle points, replacing the endpoints by infinity introduces an error which has to be bounded. Finally, termwise integration is usually justified by bounding the integral of the remainder series (and not of the first neglected term only!).

Fortunately, in many combinatorial applications, it is not difficult to validate the formal process thanks to so-called "admissibility theorems". The most famous of these is Hayman's theorem [10] which proves the first order approximation in (8) in many cases. A great advantage of Hayman's result is that the class of functions for which it applies is closed under many operations (sum, product, exponential) and contains $\exp(z)$. This makes it easy to implement this method in computer algebra systems (see [8] for applications in combinatorics). Hayman's result only covers the first order approximation, but Harris and Schoenfeld [9] gave stricter sufficient conditions that validate the full asymptotic expansion (8). A result of Odlyzko and

Richmond [12] shows that the exponential of a function from Hayman's class belongs to Harris and Schoenfeld's class. Thus in particular, the generating function $B(z)$ of the Bell numbers is admissible in their sense, which proves that (13) is an asymptotic expansion of the Bell numbers.

However, Hayman admissible functions have positive coefficients, so that $S(z)$ does not belong to this class. It turns out that a less well-known result due to Wyman [19] applies in this case. This article contains a careful analysis of a general procedure to set up a path of integration adapted to a wide class of functions. This path captures the contributions of an arbitrary (but finite) number of saddle points. The legitimation of all three approximations of the saddle-point method are reduced to checking a small set of sufficient conditions. The reason why this method is less popular than Hayman's is that testing the sufficient conditions is not as straightforward. At present, no simple closure properties have been studied that would give systematic syntactic checking easier. Using the notations of Wyman's article, we now prove that $S(z)$ is admissible in his sense, thereby concluding the proof of Theorem 1.

Wyman's admissibility proceeds in three steps to construct a contour of integration that captures the dominant parts of the integral. The first two steps are related to locating these parts and the last one deals with local expansion and approximation. We now review these steps.

2.1. **Stationary Paths and Assumption 1.** The first step of Wyman's method is an analysis of the modulus function

$$M(r,\theta)=|S(r\exp(i\theta))|.$$

The extrema of this function on a circle are solutions of

$$\frac{\partial M}{\partial\theta}=0.$$

In general, this equation defines a number of *stationary paths* with polar equations $\theta_k(r)$, $k=1,\dots,N$; this number N may change when $\partial^2M/\partial\theta^2=0$ along a path. In the special case we are considering, the stationary paths have equation

$$\tan(\theta)+\tan(r\sin\theta)=0, \tag{17}$$

or in cartesian coordinates

$$y+x\tan y=0.$$

The corresponding curves are displayed in Figure 4. Thus, this example displays a number of stationary paths that tends to infinity with r. However, since

$$|S(x+iy)|=\exp(1-e^x\cos y),\qquad x,y\in\mathbb{R}, \tag{18}$$

the values of the moduli $M_k:=M(r,\theta_k(r))$ are all bounded by $M_1=M(r,\theta_1(r))$ where $\theta_1(r)$ is the smallest positive solution of (17). This implies Assumption 1 of Wyman's which reads:

Assumption 1. *For r sufficiently close to b [the radius of convergence] there exists a continuous stationary path, with polar equation $\theta=\theta_1(r)$, by means of which we can reach the boundary $r=b$. Further, $M(r,\theta)/M_1$ is bounded uniformly in θ.*

We now prove this together with more precise information on the M_k's corresponding to maxima. We concentrate on those maxima for which $x_k>0$ since the amplitude of $|S|$ is relatively small in the left half-plane. By symmetry, it

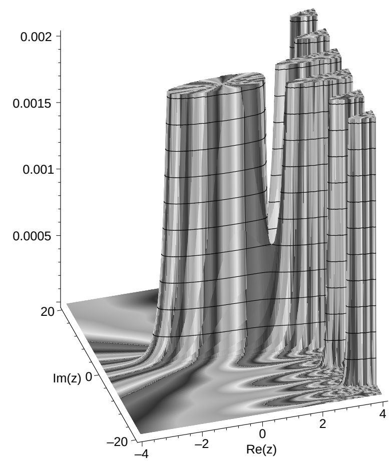

FIGURE 3. $|S(z)/z^{11}|$

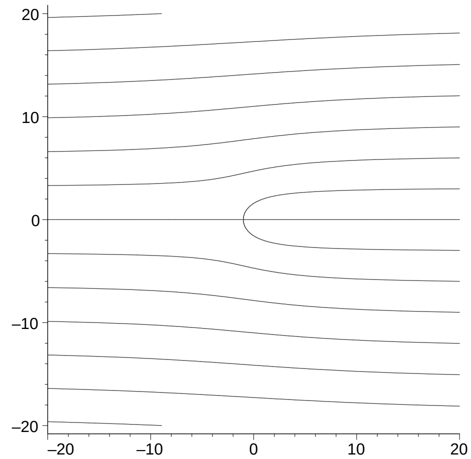

FIGURE 4. Stationary paths for $S(z)$

is then sufficient to consider $y_k > 0$. We number the stationary paths counterclockwise so that the successive maxima are given by $\theta_1, \theta_3, \ldots$. The maxima being on the same circle, this numbering implies that $x_{k+2} < x_k$ at least when $x_k > 0$. Since $-y/\tan(y)$ is an increasing function on each interval $(k\pi, (k+1)\pi)$ for $k \in \mathbb{N}$, we deduce that $y_{k+2} < 2\pi + y_k$ and therefore the ordinates of successive maxima satisfy $0 > \cos y_{k+2} > \cos y_k$. Finally, we have obtained

$$M_{k+2} < M_k, \qquad \text{when } x_k > 0,\, y_k > 0. \tag{19}$$

2.2. **Relevant Paths and Assumption 2.** Eventually, the information provided by the stationary paths will be used to construct a contour of integration. In this perspective, it is necessary to concentrate on those paths that correspond to local maxima, and among those, to select those whose contribution will not be negligible compared to the final expansion. Wyman proves that the final expansion is in descending powers of a function comparable to $\log M_1$. Therefore stationary paths $\theta_k(r)$ such that when r tends to the radius of convergence,

$$\frac{M_k(r)}{M_1(r)} = o((\log M_1(r))^a), \qquad \text{for all } a < 0$$

need not be considered. The remaining ones are called *relevant paths.*

From the inequality (19), if we show that $\theta_3(r)$ is not a relevant path, then we can conclude that the only relevant paths are $\pm\theta_1(r)$. Using the defining equation (17), we see that the first θ_k's obey

$$\frac{\sin \theta_k}{k\pi - \theta_k} = \frac{1}{r}.$$

By the implicit function theorem, θ_k is then an analytic function of $1/r$, and power series reversion shows that

$$\theta_k = k\frac{\pi}{r} - k\frac{\pi}{r^2} + \ldots, \qquad r \to \infty.$$

Simple manipulations then lead to

$$\log \frac{M_3(r)}{M_1(r)} \sim -\frac{3\pi^2}{2r} e^r.$$

Since

$$\log M_1 \sim e^r, \tag{20}$$

this is sufficient to conclude.

We now have all the elements to check whether Wyman's Assumption 2 is satisfied.

Assumption 2. *The relevant paths can be identified in such a way that the following properties are true for r sufficiently close to b.*

- (a) *The relevant paths are all continuous curves by means of which we may reach the boundary $r = b$.*
- (b) *If $\theta = \theta_k(r)$ is a relevant path then a constant $K_k > 0$ and a nonnegative integer m exist such that $M_k/M_1 \geq K_k(\log M_1)^{-m}$.*
- (c) *The number N of relevant paths is fixed and independent of r.*
- (d) *For every relevant path $\theta = \theta_k(r)$ the $\lim_{r\to b} \theta_k(r)$ exists. Further if $\theta = \theta_k(r)$ and $\theta = \theta_s(r)$ are distinct relevant paths then*

$$\lim_{r\to b} \theta_k(r) \neq \lim_{r\to b} \theta_s(r).$$

- (e) *Along every relevant path the function $M(r, \theta)$ has the property*

$$\frac{\partial^2 M}{\partial \theta^2} < 0.$$

In our case, the relevant paths are $\pm\theta_1(r)$, from which (c) follows. We have seen that θ_1 is analytic in $1/r$, this implies (a). Since $M_1 = M_{-1}$, (b) is obvious. A straightforward calculation shows that

$$\frac{\partial^2 M_1}{\partial \theta^2} = -e^{1+e^{r\cos\theta_1}\cos\theta_1} e^{r\cos\theta_1} r\left(r\cos\theta_1 + 1\right),$$

which proves (e). It is slightly unfortunate that (d) is not satisfied in our example. Indeed,

$$\lim_{r\to\infty} \theta_1(r) = \lim_{r\to\infty} -\theta_1(r) = 0.$$

However, we shall proceed with Wyman's argument and check that this assumption is unnecessary in our case.

2.3. **Behaviour Along Relevant Paths and Assumption 3.** The saddle-point analysis is concluded by locating the saddle points of the Cauchy integral (2) on the relevant paths and devising an appropriate contour of integration. For n sufficiently large, the saddle-point equation (3) has exactly one solution along each relevant path. Thus these paths can be parameterized by $Z_k(n)$ instead of $\theta_k(r)$. Conversely, denoting by $z_k(r)$ the point of modulus r on the path $\theta_k(r)$, the equation $z_1(r) = Z_1(n)$ defines a function $n(r)$.

In our case, $Z_1(n)$ is nothing but $R_+(n)$ from equation (11), which shows that

$$z_1(r) = Z_1(n) = \log n + i\pi + \cdots$$

and $n(r) \sim re^r$.

Wyman's Assumption 3 gives control over the local expansions in the neighbourhood of the saddle points.

Assumption 3. *For r sufficiently close to b each of the following are true.*

(a) *There exist positive constants p, P_1, P_2 such that*
$$P_1 \log M_1 \leq n(r) \leq P_2(\log M_1)^{1+p}.$$

(b) $Z_k(n)/z_k(r) = 1 + O(1/n)$.

(c) *In the complex w plane there exists a fixed neighbourhood $|w| \leq h$ for which the functions $\log S(Z_k(n(r)) \exp(\log M_1(r)w/n(r)))$ are all regular functions of w. Further,*
$$\lim_{r\to b} \frac{\log S\left(Z_k(n(r)) \exp(\log M_1(r)w/n(r))\right) - \log S(Z_k(n(r)))}{\log M_1(r)}$$
exists, uniformly in w, for all w within and on the boundary of $|w| \leq h$. This limit $g_k(w)$ satisfies $\Re g_k''(0) \neq 0$.

In view of our previous estimates, parts (a) and (b) are satisfied. Since S is an entire function with no zero, the first part of (c) is granted for any $h > 0$.

The previous estimates imply that
$$\lim_{r\to\infty} \frac{Z_1(n(r)) \log M_1(r)}{n(r)} = 1.$$
Therefore, for any fixed $h > 0$ and w with $|w| \leq h$, the argument of the exponential in part (c) of the assumption tends to 0 when r tends to infinity. Using the above limit several times, we finally get that
$$g_1(w) = e^w - 1$$
and the limit is uniform for any fixed $h > 0$.

2.4. **Integration.** Had Assumption 2 (d) been satisfied, the proof of Theorem 1 would end here by Wyman's result. We now proceed to follow some of the steps of Wyman's proof and check that the failure of Assumption 2 (d) is benign in our case. This illustrates that beyond his admissibility, Wyman has actually developed a general method to devise the proper contour of integration for a large class of saddle-point analyses.

In general, this contour consists of arcs of circle $|z| = |Z_k|$ joining $Z_k \exp(-i\epsilon)$ to $Z_k \exp(i\epsilon)$ (see below for the choice of ϵ), segments of lines from $Z_k \exp(\pm i\epsilon)$ to $z_k \exp(\pm i\epsilon)$ and arcs of circle $|z| = |z_k| = r(n)$ from $z_k \exp(i\epsilon)$ to $z_{k+1} \exp(-i\epsilon)$. This contour is sufficient to concentrate all terms of the asymptotic behaviour of the integral on the first type of arcs.

The infinitesimal angle ϵ is defined by
$$\epsilon = \left|\Theta^2 \log S(Z_1)\right|^{-\alpha},$$
where Θ denotes the differential operator $z\, d/dz$ and α is chosen such that
$$\frac{6p+2}{6(2p+1)} < \alpha < 1/2,$$
with p from part (a) of Assumption 3.

The purpose of part (d) of Assumption 2 is to ensure that the contour described above does not have self-intersections. In our example, the contour is a circle, $1/3 < \alpha < 1/2$ and
$$\epsilon \sim \frac{1}{(n \log n)^\alpha}.$$

In view of the location of Z_1 from (12) and (10), this value of ϵ is indeed sufficient to prevent intersections of the arcs containing Z_1 and $\overline{Z_1}$ for n sufficiently large. From there the rest of Wyman's proof applies and concludes the proof of Theorem 1.

References

[1] Beard, R. E. On the coefficients in the expansion of e^{e^t} and e^{-e^t}. *Journal of the Institute of Actuaries 76* (1950), 152–163.

[2] Bergeron, F., Labelle, G., and Leroux, P. *Combinatorial species and tree-like structures.* No. 67 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[3] Comtet, L. Inversion de $y^\alpha e^y$ et $y \log^\alpha y$ au moyen des nombres de Stirling. *Comptes-Rendus de l'Académie des Sciences 270* (1970), 1085–1088.

[4] Comtet, L. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[5] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., and Knuth, D. E. On the Lambert W function. *Advances in Computational Mathematics 5*, 4 (1996), 329–359.

[6] De Bruijn, N. G. *Asymptotic Methods in Analysis.* Dover, 1981. A reprint of the third North Holland edition, 1970 (first edition, 1958).

[7] Flajolet, P., and Odlyzko, A. M. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics 3*, 2 (1990), 216–240.

[8] Flajolet, P., and Salvy, B. Computer algebra libraries for combinatorial structures. *Journal of Symbolic Computation 20*, 5-6 (1995), 653–671.

[9] Harris, B., and Schoenfeld, L. Asymptotic expansions for the coefficients of analytic functions. *Illinois Journal of Mathematics 12* (1968), 264–277.

[10] Hayman, W. K. A generalization of Stirling's formula. *Journal für die reine und angewandte Mathematik 196* (1956), 67–95.

[11] Odlyzko, A. M. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 2*, R. Graham, M. Grötschel, and L. Lovász, Eds. Elsevier, Amsterdam, 1995, pp. 1063–1229.

[12] Odlyzko, A. M., and Richmond, L. B. Asymptotic expansions for the coefficients of analytic generating functions. *Aequationes Mathematicae 28* (1985), 50–63.

[13] Olver, F. W. J. *Asymptotics and Special Functions.* Academic Press, 1974.

[14] Salvy, B. Fast computation of some asymptotic functional inverses. *Journal of Symbolic Computation 17*, 3 (1994), 227–236.

[15] Salvy, B., and Shackell, J. Symbolic asymptotics: Multiseries of inverse functions. *Journal of Symbolic Computation 27*, 6 (June 1999), 543–563.

[16] Sirovich, L. *Techniques of Asymptotic Analysis*, vol. 2 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1971.

[17] Sloane, N. J. A., and Plouffe, S. *The Encyclopedia of Integer Sequences.* Academic Press, 1995. See also the up to date electronic version at the URL: `http://www.research.att.com/~njas/sequences/`.

[18] Uppuluri, V. R. R., and Carpenter, J. A. Numbers generated by the function $\exp(1-e^x)$. *Fibonacci Quarterly 7*, 4 (1969), 437–448.

[19] Wyman, M. The asymptotic behavior of the Laurent coefficients. *Canadian Journal of Mathematics 11* (1959), 534–555.

Algorithms Project, Inria Rocquencourt, 78153 Le Chesnay Cedex, France

E-mail address: `Bruno.Salvy@inria.fr`

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105,
78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS
Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
(France)
http://www.inria.fr
ISSN 0249-6399